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THE LIMITING DISTRIBUTION OF THE VIRTUAL WAITING TIME AND THE QUEUE SIZE FOR A SINGLE-SERVER QUEUE WITH RECURRENT INPUT AND GENERAL SERVICE TIMES

by

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1. INTRODUCTION. Suppose that in the time interval $(0, \infty)$ customers arrive at a counter at times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$. The customers are served by a single server in order of arrival. The server is idle if and only if there is no customer in the system. Denote by χ_n the service time of the n-th customer. It is supposed that the service times χ_n $(n = 1, 2, \ldots)$ and the interarrival times $\theta_n = \tau_{n+1} - \tau_n$ $(n = 1, 2, \ldots)$ are independent sequences of identically distributed, mutually independent, positive random variables with distribution functions

$$P\{\chi_n \leq x\} = H(x) \tag{1}$$

and

$$P\{\theta_n \leq x\} = F(x) . \tag{2}$$

Let $E\{\chi_n\} = \alpha$ and $E\{\theta_n\} = \beta$. Throughout this paper α and β are supposed to be finite and the trivial case $P\{\chi_n = \theta_n\} = 1$ is excluded.

Denote by $\eta(t)$ the virtual waiting time at time t, i.e., $\eta(t)$ is the time that a customer would have to wait if he arrived at time t. Let $\eta_n = \eta(\tau_n - 0)$, i.e., η_n is the actual waiting time

of the n-th arriving customer. Denote by $\xi(t)$ the queue size at time t, i.e., the total number of customers (either waiting or being served) in the system at time t. Let $\xi_n = \xi(\tau_n - 0)$, i.e., ξ_n is the queue size immediately before the arrival of the n-th customer.

In what follows we shall determine the limiting distribution of $\eta(t)$ and that of $\xi(t)$ as $t \to \infty$. We note here that the distribution of the queue size is independent of the order of service.

At this point I should like to mention briefly the idea which leads to the notion of virtual waiting time. One can suppose, without loss of generality, that each customer is assigned his service time in advance at his arrival, because the service times are identically distributed, mutually independent random variables and independent of the arrival times. Suppose that we use a reading-timer which has a clock mechanism and each time a customer arrives we set the hand forward by his future service time. Since this clock runs as long as there are customers in the system, it will at any given instant show the appropriate virtual waiting time. Thus an arriving customer can immediately see his own actual waiting time on this clock. $\eta(t)$ can also be interpreted as the occupation time of the server at time t, that is, the time that is needed to complete the service of all those customers who arrived before t. In certain queues $\eta(t)$ has a real physical meaning. For instance, if we consider reading of messages in a telegraph office, then $\eta(t)$ can be interpreted as the length of all messages which remain to be read at time t.

The process $\{\eta(t)\}$ has interest not only in the theory of queues but also in the investigation of operation of dams. If $\eta(t)$ denotes the content of a dam at time t, then $\{\eta(t)\}$ has the same

stochastic behavior as the virtual waiting time in a queueing process. (Cf. J.Gani and N.U. Prabhu [5].)

Finally we introduce the following Laplace-Stieltjes transforms:

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x) \tag{3}$$

and

$$\varphi(s) = \int_0^\infty e^{-sx}\, dF(x)$$

which are convergent if $\mathcal{R}(s) \geq 0$.

2. THE LIMITING DISTRIBUTION OF THE ACTUAL WAITING TIME.

The following results have been proved by D.V. Lindley [6]: If $\alpha < \beta$, then the limiting distribution $\lim_{n\to\infty} P\{\eta_n \leq x\} = W(x)$ exists, independent of the initial state and it is the unique solution of the following integral equation of Wiener-Hopf type

$$W(x) = \begin{cases} \int_0^\infty K(x-y)\, dW(y) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases} \tag{5}$$

where

$$K(x) = \int_0^\infty H(x+y)\, dF(y) \tag{6}$$

and further $W(0) > 0$. If $\alpha \geq \beta$ (the trivial case $P\{\chi_n = \theta_n\} = 1$ is excluded), then $P\{\lim_{n\to\infty} \eta_n = \infty\} = 1$, whence it follows that $\lim_{n\to\infty} P\{\eta_n \leq x\} = 0$ for every x irrespective of the initial state.

Define the event $\mathcal{E}$ such that $\mathcal{E}$ is said to occur at the n-th arrival if the server is found to be idle at that time. Evidently $\mathcal{E}$ is a recurrent event. If $\alpha \leq \beta$, then $\mathcal{E}$ is persistent, and if $\alpha > \beta$, then $\mathcal{E}$ is transient. (As to the theory of recurrent events we refer to W. Feller [3] pp. 278-310.)

Denote by $R(x)$ the probability that the distance between two successive occurrences of $\mathcal{E}$ is $\leq x$. If $\alpha \leq \beta$, then $R(\infty) = 1$, i.e., $R(x)$ is a proper distribution function. The mean recurrence time of $\mathcal{E}$ is

$$(7) \qquad \rho = \int_0^\infty x \, dR(x) = \int_0^\infty [1 - R(x)] \, dx = \beta / W(0) .$$

If $\alpha < \beta$, then $\rho < \infty$, whereas if $\alpha = \beta$, then $\rho = \infty$. If $\alpha > \beta$, then $R(\infty) < 1$.

Finally we note that if $F(x)$ is not a lattice distribution function, then $R(x)$ is not one either.

3. THE LIMITING DISTRIBUTION OF THE VIRTUAL WAITING TIME.

We shall prove

THEOREM 1. *If $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then the limiting distribution*

$$(8) \qquad \lim_{t \to \infty} P\{\eta(t) \leq x\} = W^*(x)$$

exists, independent of the initial state and is given by

$$(9) \qquad W^*(x) = \left(1 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} W(x) * H^*(x)$$

where $W(x)$ is defined by (5),

$$(10) \qquad H^*(x) = \begin{cases} \frac{1}{\alpha} \int_0^x [1 - H(y)] dy & \text{if } x \geq 0 , \\ 0 & \text{if } x < 0 , \end{cases}$$

and $*$ denotes convolution. If $\alpha \geq \beta$ (the trivial case $P\{\chi_n = \theta_n\} = 1$ is excluded), then $\lim_{t \to \infty} P\{\eta(t) \leq x\} = 0$ for every x, irrespective

of the initial state.

PROOF. The proof consists of two parts. First we prove that the limit exists and then we find the explicit form of the limiting distribution in case of $\alpha < \beta$. We need

LEMMA 1. Let A be an event which has the following property: if A occurs at time u and does not occur at time $u+t$, then this implies that at least one customer arrives in the interval $(u, u+t]$. Denote by $P_A(t)$ the probability that the system is in state A at time t. If $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then $\lim_{t \to \infty} P_A(t) = P_A^*$ exists and is independent of the initial state.

PROOF. Denote by $M(t)$ the expected number of occurrences of $\mathcal{E}$ in the time interval $(0,t]$. Let $Q_A^*(t)$ denote the probability that the system is in state A at time t and $\mathcal{E}$ never occurs in the interval $(0,t]$. Measuring time from an occurrence of $\mathcal{E}$ denote by $Q_A(t)$ the probability that A occurs at time t and $\mathcal{E}$ never occurs during the interval $(0,t]$. Evidently

$$(11) \qquad P_A(t) = Q_A^*(t) + \int_0^\infty Q_A(t-u)\, dM(u).$$

If $\alpha < \beta$, then $\mathcal{E}$ is a persistent event and consequently $\lim_{t\to\infty} Q_A^*(t) = \lim_{t\to\infty} Q_A(t) = 0$. We shall prove later that $Q_A(u)$ is of bounded variation in every finite interval $[0,t]$. If $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then by a theorem of D. Blackwell [1]

we have for all $h > 0$ that

(12) $$\lim_{t \to \infty} \frac{M(t+h) - M(t)}{h} = \frac{1}{\rho},$$

where ρ is defined by (7). Thus if $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then it follows from (11) and (12) that

(13) $$\lim_{t \to \infty} P_A(t) = \frac{1}{\rho} \int_0^\infty Q_A(u)\, du$$

irrespective of the initial state. (Cf. W.L. Smith [9], and [11] pp. 227-228.) Since

(14) $$Q_A(u) \leq 1 - R(u)$$

for $u \geq 0$, the integral on the right hand side of (13) converges.

It remains to prove that $Q_A(u)$ is of bounded variation in any finite interval $[0,t]$. The proof is based on an idea of W.L. Smith [10]. Measure time from an occurrence of $\mathcal{E}$. Denote by $\nu(t)$ the number of arrivals in the interval $(0,t]$. Define $\chi_t = 1$ if A occurs at time t and $\mathcal{E}$ does not occur in the interval $(0,t]$; $\chi_t = 0$ otherwise. For $0 \leq u \leq t$ we have

(15) $$|Q_A(t) - Q_A(u)| \leq E\{\chi_t - \chi_u\} + 2\,E\{\nu(t) - \nu(u)\},$$

for,

$$Q_A(t) - Q_A(u) = E\{\chi_t - \chi_u\} = P\{\chi_u = 0,\ \chi_t = 1\} - P\{\chi_u = 1,\ \chi_t = 0\},$$

whence

$$|Q_A(t) - Q_A(u)| \leq E\{\chi_t - \chi_u\} + 2\,P\{\chi_u = 1,\ \chi_t = 0\}$$

and by assumption

$$P\{\chi_u = 1, \ \chi_t = 0\} \le P\{\nu(t) - \nu(u) \ge 1\} \le E\{\nu(t) - \nu(u)\} .$$

Accordingly for any subdivision $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0,t]$

$$(16) \quad \sum_{k=1}^{n} |Q_A(t_k) - Q_A(t_{k-1})| \le E\{\chi_t - \chi_0\} + 2\, E\{\nu(t)\} \le 1 + 2\, E\{\nu(t)\}$$

Since $E\{\nu(t)\}$ is finite for every $t \ge 0$ ($E\{\nu(t)\} \le Ct+1$ where C is a positive constant), it follows that $Q_A(t)$ is of bounded variation in $[0,t]$. This completes the proof of Lemma 1.

REMARK 1. Let $\alpha < \beta$ and $F(x)$ be a non-lattice distribution function. Consider a monotone non-decreasing sequence of events $A_1 \subset A_2 \subset \ldots \subset A_k \subset \ldots$ for which A_k satisfies the assumptions of Lemma 1 and $\lim_{k \to \infty} A_k = \Omega$, the sure event. Let $\lim_{t \to \infty} P_{A_k}(t) = P^*_{A_k}$ defined by (13). Then $\lim_{k \to \infty} P^*_{A_k} = 1$. For, in this case $Q_{A_k}(u)$ $(k = 1, 2, \ldots)$ is a monotone non-decreasing sequence and $\lim_{k \to \infty} Q_{A_k}(u) = 1 - R(u)$, whence by Beppo Levi's theorem (Cf. F. Riesz and B. Sz-Nagy [8] p.34)

$$\lim_{k \to \infty} \int_0^{\infty} Q_{A_k}(u)\, du = \int_0^{\infty} [1 - R(u)]\, du = \rho .$$

The statement follows from (13).

Now we shall prove that if $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then the limit (8) exists and is independent of the initial state. Define A as the event that the virtual waiting time is $\le x$, where $x \ge 0$. Then the event

A satisfies the assumptions of Lemma 1 and $P_A(t) = P\{\eta(t) \leq x\}$. Thus by Lemma 1 the limit (8) exists and $W^*(x)$ is a monotone non-decreasing function of x and by Remark 1 $W^*(\infty) = 1$.

If $\alpha > \beta$, then $\lim_{t \to \infty} P\{\eta(t) \leq x\} = 0$ for every x irrespective of the initial state. Since $\eta(t) \geq \eta_n$ for $\tau_{n-1} < t < \tau_n$, $n = 1, 2, \ldots$, and by Lindley's theorem $P\{\lim_{n\to\infty} \eta_n = \infty\} = 1$ for $\alpha \geq \beta$, we can conclude also that in this case $P\{\lim_{t\to\infty} \eta(t) = \infty\} = 1$.

REMARK 2. Now we shall prove directly that $P\{\lim_{t\to\infty} \eta(t) = \infty\} = 1$ if $\alpha > \beta$. Denote by $\nu(t)$ the number of arrivals in the interval $(0,t]$. By a theorem of J.L. Doob [2] we have

$$P\left\{\lim_{t\to\infty} \frac{\nu(t)}{t} = \frac{1}{\beta}\right\} = 1. \tag{17}$$

since obviously

$$\eta(t) \geq \eta(0) + \sum_{i=1}^{\nu(t)} \chi_i - t,$$

we have

$$\frac{\eta(t)}{t} \geq \frac{\eta(0)}{t} + \frac{\nu(t)}{t} \frac{1}{\nu(t)} \sum_{i=1}^{\nu(t)} \chi_i - 1. \tag{18}$$

If $t \to \infty$ in (18), then we have with probability one that $\eta(0)/t \to 0$, $\nu(t)/t \to 1/\beta$ and

$$\frac{1}{\nu(t)} \sum_{i=1}^{\nu(t)} \chi_i \to \alpha.$$

The latter follows from an easy extension of the strong law of large numbers. Thus by (18)

$$\lim_{t\to\infty} \inf \frac{\eta(t)}{t} \geq \frac{\alpha}{\beta} - 1 > 0$$

with probability one, whence

(19) $$P\{\lim_{t\to\infty} \eta(t) = \infty\} = 1 .$$

This proves that if $\alpha > \beta$, then $\lim_{t\to\infty} P\{\eta(t) \leq x\} = 0$ for every irrespective of the initial state.

Finally it remains only to find $\lim_{t\to\infty} P\{\eta(t) \leq x\} = W^*(x)$ if $\alpha < \beta$ and $F(x)$ is not a lattice distribution function. First we define a random variable $\theta(t)$ as the time between t and the first arrival after t. Then we observe that the vector variables $\{\eta(t), \theta(t)\}$ form a Markov process. The initial state is given by $(\eta(0),\theta(0))$ where $\eta(0)$ is the initial occupation time of the server and $\theta(0) = \tau_1$ is the time of the first arrival. (We note that if the input is a Poisson process, then $\{\eta(t)\}$ is a Markov process in itself.) Define now A as follows: A occurs at time t if $\eta(t) \leq x$ and $\theta(t) \leq y$ where $x \geq 0$ and $y \geq 0$. This A satisfies the assumptions of Lemma 1 and if $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then by Lemma 1 we can conclude that

(20) $$\lim_{t\to\infty} P\{\eta(t) \leq x, \theta(t) \leq y\} = W^*(x,y)$$

exists and is independent of the initial state. $W^*(x,y)$ is a two dimensional distribution function, because by Remark 1 $W^*(\infty,\infty) = 1$.

Let

(21) $$\Omega^*(s,w,t) = E\{e^{-s\,\eta(t) - w\,\theta(t)}\}$$

and

(22) $$\Omega^*(s,w) = \int_0^\infty \int_0^\infty e^{-sx-wy}\, d^2_{xh}\, W^*(x,y) .$$

If $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then by

(20)

(23) $$\lim_{t\to\infty} \Omega^*(s,w,t) = \Omega^*(s,w)$$

for $\mathcal{R}(s) \geq 0$ and $\mathcal{R}(w) \geq 0$. If

(24) $$\Omega^*(s) = \int_0^\infty e^{-sx} \, dW^*(x) \quad ,$$

then obviously $\Omega^*(s) = \Omega^*(s,0)$.

Now we shall prove

LEMMA 2. Denote by $m(t)$ the expected number of arrivals in the time interval $(0,t]$. If $m(t+\Delta t) - m(t) = 0(\Delta t)$, then

(25) $$\frac{\Omega^*(s,w,t+\Delta t) - \Omega^*(s,w,t)}{\Delta t} = (w+s)\Omega^*(s,w,t) - sP_0(t)\Phi(w,t) -$$

$$\frac{m(t+\Delta t) - m(t)}{\Delta t}\left[1 - \psi(s)\varphi(w)\right]\Omega(s,t) + \frac{o(\Delta t)}{\Delta t}$$

where $P_0(t) = P\{\eta(t) = 0\}$, $\Omega(s,t) = E\{e^{-s\,\eta(t)} \mid \theta(t) = 0\}$

and $\Phi(w,t) = E\{e^{-w\theta(t)} \mid \eta(t) = 0\}$.

PROOF. if $\theta(t) > \Delta t$, then $\theta(t+\Delta t) = \theta(t) - \Delta t$ and $\eta(t+\Delta t) = \max\,(0, \eta(t) - \Delta t)$. Thus

$$E\{e^{-s\,\eta(t+\Delta t) - w\,\theta(t+\Delta t)} \mid \theta(t) > \Delta t\} =$$

$$P\{\eta(t) = 0, \theta(t) > \Delta t\}\; E\{e^{-s\,\eta(t)-w\theta(t)} \mid \eta(t)=0, \theta(t) > \Delta t\}\,(1+w\Delta t) +$$
$$P\{0 < \eta(t) \leq \Delta t, \theta(t)>\Delta t\} E\{e^{-s\eta(t)-w\theta(t)} \mid \eta(t) > \Delta t, \theta(t) > \Delta t\}\,[1+(w+s)\Delta t] + 0(\Delta t)$$

If $\theta(t) \leq \Delta t$, then $\theta(t+\Delta t) = \theta - \epsilon_1\Delta t$ and $\eta(t+\Delta t) = \eta(t) + \chi - \epsilon_2\Delta t$ where χ is the total service time of all those customers who arrive in the interval $(t, t+\Delta t]$, θ is the interarrival time between the last arrival in $(t, t+\Delta t]$ and the first arrival after

$t+\Delta t$, and further $0 \leq \epsilon_1 \leq 1$ and $0 \leq \epsilon_2 \leq 1$. Thus

$$E\{e^{-s\eta(t+\Delta t)-w\theta(t+\Delta t)} | \theta(t) \leq \Delta t\} = \psi(s)\varphi(w)\, E\{e^{-s\eta(t)-w\theta(t)} | \theta(t) \leq \Delta t\} + o(\Delta t).$$

Since $P\{\theta(t) \leq \Delta t\} \leq m(t+\Delta t) - m(t) = O(\Delta t)$ we obtain by the theorem of total expectation that

$$\Omega^*(s,w,t+\Delta t) = [1+(w+s)\Delta t]\,\Omega^*(s,w,t) - s\Delta t\, P\{\eta(t) = 0, \theta(t) > t\}.$$

(26) $$E\{e^{-w\theta(t)} | \eta(t) = 0, \theta(t) > \Delta t\} - [1 - \psi(s)\varphi(w)]E\{e^{s\eta(t)} | \theta(t) \leq \Delta t\}\,.$$
$$P\{\theta(t) \leq \Delta t\} + o(\Delta t)\,.$$

Since $m(t+\Delta t) - m(t) = P\{\theta(t) \leq \Delta t\} + o(\Delta t)$ also holds, we get finally

$$\Omega^*(s,w,t+\Delta t) = [1 + (w+s)\Delta t]\,\Omega^*(s,w,t) - s\Delta t\, P\{\eta(t) = 0\}\,.$$

(27) $$E\{e^{-w\theta(t)} | \eta(t) = 0\} - [1 - \psi(s)\varphi(w)]\, E\{e^{-s\eta(t)} | \theta(t) = 0\}\,.$$
$$[m(t+\Delta t) - m(t)] + o(\Delta t)\,,$$

which is in agreement with (25).

By Lemma 1 the following limits exist $\lim_{t\to\infty} \Omega^*(s,w,t) = \Omega^*(s,w)$ (cf. (23)), $\lim_{t\to\infty} P_0(t) = P_0^*$ (It is easy to prove directly that $P_0^* = 1 - \alpha/\beta$. (Cf. [11] p. 142.) and $\lim_{t\to\infty} \Phi(w,t) = \Phi(w)$, say.

By Lindley's theorem $\lim_{t\to\infty} \Omega(s,t) = \Omega(s)$ where

(28) $$\Omega(s) = \int_0^\infty e^{-sx}\, dW(x)$$

and $W(x)$ is defined by (5). By Blackwell's theorem

(29) $$\lim_{t\to\infty} \frac{m(t+\Delta t) - m(t)}{\Delta t} = \frac{1}{\beta}\,.$$

If we let $t \to \infty$ in (25) we obtain

(30) $$(w+s)\Omega^*(s,w) = sP_0^*\Phi(w) + \frac{[1-\Psi(s)\Phi(w)]}{\beta}\Omega(s) .$$

If $w \to 0$ in (30), then we get

(31) $$\Omega^*(s) = P_0^* + \frac{[1-\Psi(s)]}{\beta s}\Omega(s) .$$

Since $\Omega^*(0) = 1$, we obtain that $P_0^* = 1 - \alpha/\beta$. Thus finally the Laplace-Stieltjes transform of $W^*(x)$ is given by

(32) $$\Omega^*(s) = (1 - \frac{\alpha}{\beta}) + \frac{\alpha}{\beta}\frac{[1-\Psi(s)]}{\alpha s}\Omega(s) ,$$

whence (9) follows by inversion. This completes the proof of Theorem 1.

EXAMPLES. (i) Suppose that $F(x) = 1 - e^{-\lambda x}$ $(x \geq 0)$ and $H(x)$ is arbitrary. In this case $\beta = 1/\lambda$. If $\lambda\alpha < 1$, then

(33) $$\Omega(s) = \frac{1-\lambda\alpha}{1-\lambda\frac{1-\Psi(s)}{s}}$$

and thus by (32) $\Omega^*(s) = \Omega(s)$, i.e., $W^*(x) = W(x)$.

(ii) Suppose that $F(x)$ is arbitrary and $H(x) = 1 - e^{-\mu x}$ $(x \geq 0)$. In this case $\alpha = 1/\mu$. If $\mu\beta > 1$, then

(34) $$\Omega(s) = (1-\delta) + \delta\frac{\mu(1-\delta)}{\mu(1-\delta)+s}$$

where $z = \delta$ is the only root of $z = \varphi(\mu(1-z))$ inside the unit circle. Now if we suppose that $F(x)$ is not a lattice distribution function, then we obtain by (32) that

(35) $$\Omega^*(s) = (1 - \frac{1}{\mu\beta}) + \frac{1}{\mu\beta}\frac{\mu(1-\delta)}{\mu(1-\delta)+s} .$$

From (34)

(36) $$W(x) = 1 - \delta e^{-\mu(1-\delta)x} \quad \text{if} \quad x \geq 0,$$

and from (35)

$$W^*(x) = 1 - \frac{1}{\mu\beta} e^{-\mu(1-\delta)x} \quad \text{if} \quad x \geq 0 . \tag{37}$$

4. THE LIMITING DISTRIBUTION OF THE QUEUE SIZE. The following two theorems are concerned with the limiting distribution of the queue size. Formulas (39) and (47) were first found by Mr. M. Aczél (oral communication made in January 1958 during a meeting on "Queueing Theory and Practice" arranged by the Institute for Engineering Production of the University of Birmingham, England). Formula (47) has also been proved by T. Kawata [7]. Under some restrictive conditions the existence of the limiting distribution of the queue size for many-server queues has been investigated by P.D. Finch [4].

THEOREM 2. *If* $\alpha < \beta$ *and* $F(x)$ *is not a lattice distribution function, then the limiting distribution*

$\lim_{t\to\infty} P\{\xi(t) = k\} = P_k^*$ $(k = 0, 1, \ldots)$ *exists and is independent of the initial queue size. We have*

$$P_0^* = 1 - \frac{\alpha}{\beta} \tag{38}$$

and for $k = 1, 2, \ldots$

$$P_k^* = \frac{\alpha}{\beta} \int_0^\infty [F_{k-1}(x) - F_k(x)]\, d[W(x) * H^*(x)] \tag{39}$$

where $F_k(x)$ *denotes the k-th iterated convolution of* $F(x)$ *with itself;* $F_0(x) = 1$ *if* $x \geq 0$, *and* $F_0(x) = 0$ *if* $x < 0$; $W(x)$ *is defined by* (5); *and* $H^*(x)$ *is defined by (10). If* $\alpha \geq \beta$, *then* $\lim_{t\to\infty} P\{\xi(t) = k\} = 0$ $(k = 0, 1, \ldots)$ *irrespective of the initial queue size.*

PROOF. If we define A as the event that the queue size is $\leq k$ $(k = 0, 1, \ldots)$, then this event satisfies the conditions of Lemma 1. Accordingly if $\alpha < \beta$ and $F(x)$ is not a lattice distribution function, then the limiting distribution $\lim_{t \to \infty} P\{\xi(t) \leq k\}$ exists and is independent of the initial state. If $\alpha \geq \beta$, then $\lim_{t \to \infty} P\{\xi(t) \leq k\} = 0$ $(k = 0, 1, \ldots)$. This follows from (50) which will be proved later.

REMARK 3. We shall give a direct proof for the case $\alpha > \beta$. Denote by $\delta(t)$ the number of departures in the time interval $(0, t]$ and by $\nu(t)$ the number of arrivals in $(0, t]$. Then $\xi(t) = \xi(0) + \nu(t) - \delta(t)$, whence

$$(40) \qquad \frac{\xi(t)}{t} = \frac{\xi(0)}{t} + \frac{\nu(t)}{t} - \frac{\delta(t)}{t} .$$

By Doob's theorem $\lim_{t \to \infty} \nu(t)/t = 1/\beta$ and $\limsup_{t \to \infty} \delta(t)/t \leq 1/\alpha$ with probability one. Since $\lim_{t \to \infty} \xi(0)/t = 0$ with probability 1 we obtain from (40) that

$$\liminf_{t \to \infty} \frac{\xi(t)}{t} \geq \frac{1}{\beta} - \frac{1}{\alpha} > 0$$

with probability one, i.e.,

$$(41) \qquad P\{\lim_{t \to \infty} \xi(t) = \infty\} = 1 .$$

To find P_k^* for $k = 1, 2, \ldots$ we can write that

$$P\{\xi(t) = k\} = Q_k^*(t) + \int_0^t [F_{k-1}(t-u) - F_k(t-u)] \int_0^{t-u} [1 - H(t-u-y)] .$$

$$(42) \qquad d_y P\{\eta(u) \leq y \mid \theta(u) = 0\} \, d\, m(u) ,$$

where $Q_k^*(t)$ is the probability that the queue size is k at time t and there is no arrival in the time interval $(0,t]$. The second term on the right hand side of (42) can be obtained in the following way: The customer being served at time t arrives at time u $(0 \leq u \leq t)$, his waiting time is y $(0 \leq y \leq t-u)$ and in the interval $(u,t]$ $k-1$ customers arrive. If $\alpha < \beta$ and $F(x)$ is not a lattice distribution function then $\lim_{u \to \infty} P\{\eta(u) \leq y \mid \theta(u) = 0\} = W(y)$ where $W(y)$ is defined by (5). By using (29) we obtain from (42) that $\lim_{t \to \infty} P\{\xi(t) = k\} = P_k^*$ where P_k^* is given by (39) for $k = 1, 2, \ldots$. If $k = 0$, then $P_0^* = 1 - \alpha/\beta$, because

$$(43) \qquad \lim_{t \to \infty} P\{\xi(t) = 0\} = 1 - \sum_{k=1}^{\infty} P_k^* = 1 - \frac{\alpha}{\beta} .$$

REMARK 4. If

$$(44) \qquad W_1 = \int_0^{\infty} x \, dW(x)$$

is finite, then

$$(45) \qquad \sum_{k=0}^{\infty} k \, P_k^* = \frac{1}{\beta} (W_1 + \alpha) .$$

An intuitive proof is as follows: Obviously

$$\int_0^t \xi(u) \, du - \sum_{i=1}^{\nu(t)} (\eta_i + \chi_i)$$

is bounded with probability 1, i.e.,

$$(46) \qquad \lim_{t \to \infty} \frac{1}{t} \int_0^t \xi(u) \, du = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{\nu(t)} (\eta_i + \chi_i) = \frac{1}{\beta} (W_1 + \alpha)$$

with probability one. For, by (17) $\nu(t)/t \to 1/\beta$ with probability 1, by the strong law of large numbers

$$\lim_{t\to\infty} \frac{1}{\nu(t)} \sum_{i=1}^{\nu(t)} \chi_i = \alpha$$

with probability 1, and by the ergodic theorem

$$\lim_{t\to\infty} \frac{1}{\nu(t)} \sum_{i=1}^{\nu(t)} \eta_i = W_1$$

with probability 1. If we suppose that $\{\xi(t)\}$ is a stationary process, then evidently

$$E\{\xi(t)\} = \sum_{k=0}^{\infty} k\, P_k^*$$

for every $t \geq 0$, and if we form the expectation of (46) we obtain (45).

THEOREM 3. <u>If</u> $\alpha < \beta$, <u>then</u> $\lim_{n\to\infty} P\{\xi_n = k\} = P_k$ $(k=1,2, \ldots)$ <u>exists independent of the initial queue size and we have</u>

(47) $$P_k = \int_0^{\infty} [F_k(x) - F_{k+1}(x)]\, d[W(x) * H(x)]$$

<u>where</u> $W(x)$ <u>is defined by</u> (5). <u>If</u> $\alpha \geq \beta$, <u>then</u> $\lim_{n\to\infty} P\{\xi_n = k\} = 0$ <u>for every</u> k <u>irrespective of the initial queue size</u>.

PROOF. The event $\xi_{n+k+1} \leq k$ occurs if and only if the n-th arriving customer departs before the n+k+1 st customer arrives, i.e., if and only if the queue size immediately after the departure of the n-th arriving customer is $\leq k$. Thus for arbitrary initial queue size $\xi(0)$ we have

(48) $$P\{\xi_{n+k+1} \leq k\} = \int_0^{\infty} [1 - F_{k+1}(x)]\, d[W_n(x) * H(x)]$$

where $W_n(x) = P\{\eta_n \leq x\}$, because the queue size immediately after the departure of the n-th arriving customer is equal to the number of arrivals during the waiting time and the service time of the n-th arriving customer.

If $\alpha < \beta$, then $\lim_{n \to \infty} W_n(x) = W(x)$ and by (48)

(49) $$\lim_{n \to \infty} P\{\xi_n \leq k\} = \int_0^\infty [1 - F_{k+1}(x)] d[W(x) * H(x)]$$

which proves (47).

If $\alpha \geq \beta$, then $\lim_{n \to \infty} W_n(x) = 0$ for every x and by (48)

(50) $$\lim_{n \to \infty} P\{\xi_n \leq k\} = 0$$

for every k.

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